Example Problems - Electric Field

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1 Continuous Lines of Charge

1.1 Problem - Uniform, Straight, Line of Charge

A charge Q is uniformly distributed over the y axis, from $y = a$ to $y = b$. Determine the electric field at any point on the x-axis, $x \neq 0$.

1.2 Solution

We determine the electric field due to the line charge by breaking it up into very small pieces, determining the electric field due to each piece, and adding them together. We will break the line charge up into small enough pieces such that they can each be considered a point charge, and we know what the electric field due to a point charge is:

$$\vec{E} = \frac{kq}{r^2}\hat{r}. \tag{1}$$

The procedure for doing this is to construct an integral that must be evaluated and then integrating. We must setup this integral such that the integration variable indicates what piece we are looking at.

The total charge in the line is Q, and this is uniformly distributed over a length $b - a$, so the charge density is a constant, $\lambda = \frac{Q}{b-a}$. Now consider the small piece of charge sitting at $(0, y')$. The variable y' is a parameter that we can use to uniquely identify each point on the line charge, every point on the line charge corresponds to some value for y' between a and b, so this will be our integration variable. We must write everything that depends on y' in terms of y', this part is essential.

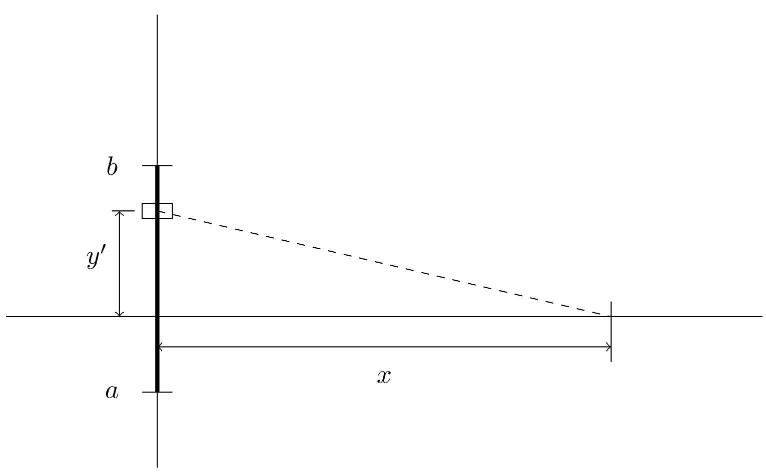

Figure 1: Illustration of the line charge.

First, we determine the amount of charge in this piece. We are assuming that the piece is small enough to be considered a point charge, so it is perfectly reasonable to say that it is an “infinitesimal length” of charge, which we would then write as dy' since y' is our integration variable. The amount of charge in the piece of charge at y' is its length times the charge density at $y = y'$: $\lambda dy'$.

We also need to know 1) how far this piece of charge is from the observation point, and 2) the direction of the line pointing from this piece of charge to the observation point. These are required by Eq 1 (r and $\hat{r}$). We will do this by first determine the displacement vector from the piece of charge to the observation point.

Let $\vec{r}_s$ and $\vec{r}_o$ be the position vectors of the charge piece and observation point respectively. We then have

$$\vec{r}_s = y'\hat{j} \tag{2}$$

$$\vec{r}_o = x\hat{i} \tag{3}$$

$$\vec{\Delta r} = \vec{r}_o - \vec{r}_s = x\hat{i} - y'\hat{j} \tag{4}$$

$$r = \left|\vec{\Delta r}\right| = \sqrt{x^2 + y'^2} \tag{5}$$

$$\hat{r} = \frac{\vec{\Delta r}}{|\Delta r|} = \frac{x\hat{i} - y'\hat{j}}{\sqrt{x^2 + y'^2}} \tag{6}$$

The electric field can now be determined from Eq 1. Since this is not the net field, but only the contribution to the net field from the piece of charge at y', it is common to refer to this as $d\vec{E}$ because it implies that there is still an integration to be performed. We have,

$$d\vec{E} = \frac{k\lambda dy'}{x^2 + y'^2} \frac{x\hat{i} - y'\hat{j}}{\sqrt{x^2 + y'^2}} = \frac{k\lambda(x\hat{i} - y'\hat{j})}{(x^2 + y'^2)^{3/2}} dy' \tag{7}$$

We must remember that λ is not given directly in the problem, so we must replace it with $Q/(b - a)$ in our final answer.

The field in Eq 7 is the field due to only the piece of charge at $(0, y')$. The net field will be the sum of the field due to all charge pieces making up the charge line. To determine the net field, we just need to evaluate Eq 7 for all values of y' between a and b and add them all together. This is exactly what integration does. The net field is

$$\vec{E}(x, 0) = \int_a^b \frac{k\lambda(x\hat{i} - y'\hat{j})}{(x^2 + y'^2)^{3/2}} dy' \tag{8}$$

Recall that $\int f(x) + g(x)dx = \int f(x)dx + \int g(x)dx$, so we can work on the x and y components of the field separately,

$$\vec{E}(x, 0) = \int_a^b \frac{k\lambda x}{(x^2 + y'^2)^{3/2}} dy'\hat{i} + \int_a^b \frac{-k\lambda y'}{(x^2 + y'^2)^{3/2}} dy'\hat{j} \tag{9}$$

1.2.1 y-component

Let’s look at the y component first,

$$E_y = -k\lambda \int_a^b \frac{y'}{(x^2 + y'^2)^{3/2}} dy'. \tag{10}$$

The integration can be performed simply using a variable substitution. Let $u^2 = x^2 + y'^2$. We must replace all occurrences of y' in the integral with u and change the integration limits to correspond to u.

$$d(u^2) = 2udu = 2y'dy' \tag{11}$$

$$udu = y'dy' \tag{12}$$

$$(x^2 + y'^2)^{3/2} = u^3 \tag{13}$$

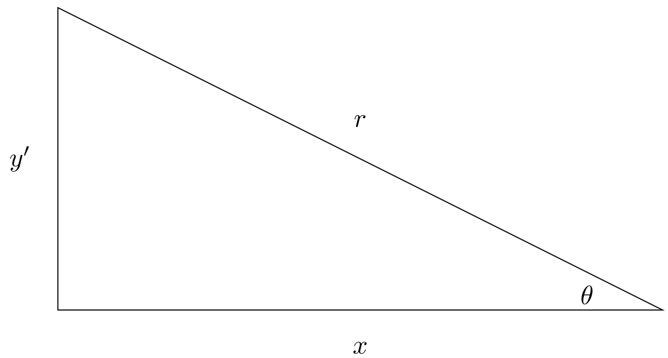

Figure 2: Geometric representation of the variable substitution $y' = x \tan\theta$

Performing the substitution gives

$$E_y = -k\lambda \int_{\sqrt{x^2+a^2}}^{\sqrt{x^2+b^2}} \frac{udu}{u^3} dy' = -k\lambda \int_{\sqrt{x^2+a^2}}^{\sqrt{x^2+b^2}} \frac{du}{u^2}. \tag{14}$$

This integration can be performed easily,

$$E_y = -k\lambda \left.\frac{-1}{u}\right|_{\sqrt{x^2+a^2}}^{\sqrt{x^2+b^2}} = k\lambda \left(\frac{1}{\sqrt{x^2+b^2}} - \frac{1}{\sqrt{x^2+a^2}}\right) = \frac{kQ}{(b-a)} \left(\frac{1}{\sqrt{x^2+b^2}} - \frac{1}{\sqrt{x^2+a^2}}\right) \tag{15}$$

We can check that Eq 15 is consistent with what we expect. First, if $|b| > |a|$, then the y-component will be negative. If $|b| = |a|$ it is zero, and if $|b| < |a|$, it is positive. This is exactly what we would expect for a positive line charge.

1.2.2 x-component

Now consider the x-component. We need to evaluate the integral,

$$E_x = k\lambda x \int_a^b \frac{1}{(x^2 + y'^2)^{3/2}} dy'. \tag{16}$$

Again, this integral can be performed simply with a variable substitution, but we cannot use the same substitution that was used for the y-component. Instead, the substitution $y' = x \tan\theta$ will work. At this point, the substitution may seem a little arbitrary, and from a pure mathematical perspective, it pretty much is. We choose this substitution because it is the one that works, and we know that it is the one that works because we try different substitutions until one works. However, this substitution just results in a change of our integration variable, and therefore θ must just be a parameter that uniquely identifies a piece of the line charge. Let's take a moment to understand just what the substitution means, physically, which will end up being useful when actually performing the substitution.

The substitution $y' = x \tan\theta$ corresponds to specifying the charge piece that we are computing the electric field for as the angle between the line going from the charge piece to the observation point and the x-axis. Figure 2 shows the right triangle that relates the relative parameters. From this triangle, we see that:

$$\sin\theta = y'/r, \tag{17}$$

$$\cos\theta = x/r, \tag{18}$$

$$\tan\theta = y'/x. \tag{19}$$

Using these relations, we can easily perform the substitution, remembering that we must completely remove y' from the integral.

$$dy' = x\sec^2\theta d\theta = \frac{x}{\cos^2\theta}d\theta, \tag{20}$$

$$(x^2+y'^2)^{3/2} = r^3 = \frac{x^3}{\cos^3\theta}, \tag{21}$$

$$E_x = k\lambda x\int\frac{1}{(x^2+y'^2)^{3/2}}dy' = k\lambda x\int\frac{\cos^3\theta}{x^3}\frac{x}{\cos^2\theta}d\theta \tag{22}$$

$$= \frac{k\lambda}{x}\int\cos\theta d\theta. \tag{23}$$

We still need to determine the limits of integration for this variable substitution, but we will use a trick here. We know that the lower limit on the integration must correspond to the bottom of the charge line at $y = a$. This will just be some angle that we could in principle calculate, but in this particular case we will not need to. The same is true for the upper limit. Let's denote these angles as θ_{min} and θ_{max}. Then,

$$E_x = \frac{k\lambda}{x}\int_{\theta_{\text{min}}}^{\theta_{\text{max}}}\cos\theta d\theta = \frac{k\lambda}{x}\sin\theta\Big|_{\theta_{\text{min}}}^{\theta_{\text{max}}} = \frac{k\lambda}{x}\left(\sin\theta_{\text{max}} - \sin\theta_{\text{min}}\right) \tag{24}$$

Since we know that θ_{max} physically corresponds to the top end of the line charge, we can directly determine $\sin\theta_{\text{max}}$ using the appropriate triangle. Considering Figure 2 with $y' = b$ gives $\sin\theta_{\text{max}} = b/r = b/\sqrt{x^2+b^2}$. We can determine $\sin\theta_{\text{min}}$ in the same way (but note that θ_{min} will be a negative angle), and we finally have

$$E_x = \frac{k\lambda}{x}\left(\frac{b}{\sqrt{x^2+b^2}} + \frac{a}{\sqrt{x^2+a^2}}\right) = \frac{kQ}{x(b-a)}\left(\frac{b}{\sqrt{x^2+b^2}} + \frac{a}{\sqrt{x^2+a^2}}\right) \tag{25}$$

We now have the full electric field for an arbitrary point on the x axis.